

Young Children's Early Deductive Reasoning in Number: A Dialogic and Linguistic Approach

Carol Murphy¹⁾

The University of Waikato, New Zealand

Abstract

In this paper I present an examination of three six year-old children's interaction with a task intended to encourage reasoning and collaboration in number. A case is made for the importance of deductive reasoning in supporting inductive reasoning and for the potential of hypothetical deductive reasoning in supporting concept reification in early number. The children's discourse is analysed using a framework based on opinion/belief, plausibility and deductive reasoning schema in relation to the functional use of actuality and modality linguistic terms. The analysis suggests that the children were able to transition to deductive reasoning and this was reflected in their discourse through a shift to modality, and that this shift suggested a sense of authority by the children in validating their thinking.

Keywords : deductive reasoning, inductive reasoning, discourse, mathematics, number relationships, early childhood

Corresponding author, ¹⁾carolmm@waikato.ac.nz

Introduction

The making of assertions through logic and reasoning is intrinsic to mathematics in generating statements of truth and in developing rigorous and structured arguments related to proof (Stylianidis, 2007). Various mathematics curricula have recognised the importance of reasoning, and the justification of ideas has been a key element of school mathematics for some time (Battista & Clements, 1995). Nevertheless early years mathematics education has often focused on problem solving; logical reasoning and proof have been less well represented. With regard to Schoenfeld's (2009) statement, if problem solving is the "heart of mathematics," then "proof is its soul" (p.xii), it would seem desirable that pre-proof experiences, as legitimate initial approaches to proof, are included alongside problem solving in early years mathematics education.

The importance of pre-proof experiences has been considered at secondary school level. For example, McFeetors and Mason's (2009) study explored the use of a logic game to encourage students to make chains of logical reasoning and convincing arguments. But less research has been carried out into pre-proof experiences in early years mathematics. Developmental constraints on young children's reasoning associated with Piaget and Inhelder (1969) may have influenced early years' educators' views on such experiences. In Piagetian theory, the construction of formal logic is seen as a final step in children's development (formal operations) that happens around twelve years of age. Hence young children's thinking is thought to be non-reflective, unsystematic and illogical (Battista & Clements, 1995; Johnson-Laird, 2008). Furthermore, for lower attaining children, a limited knowledge base in number work may also be seen as a constraint to reasoning in arithmetic. Nevertheless Diezmann, Watters, and English (2002) found that reasoning could be encouraged with young children and that constraints were more likely to come from teacher expectations, the nature of classroom discourse, and limited authentic opportunities. Markovits (2013) has also shown that young children are capable of logical responses.

This paper explores the potential of a task to provide an authentic opportunity for young primary school children (6 year olds) to develop logical deductive reasoning as a pre-proof experience. The three children had a limited knowledge base in arithmetic and were mainly reliant on counting strategies to solve addition and subtraction problems. In examining the

children's engagement with the logical reasoning of a task that involved number comparison relationships, the paper considers if such a task has the potential to bring together proof and problem-solving in early years mathematics education.

Deductive and inductive reasoning in mathematics

Inductive reasoning relates to generalisations and conjectures based on evidence found in physical world observations in seeing a pattern. The reasoning is from the particular to the general. Deductive reasoning, on the other hand, starts by defining a particular abstract premise that is already known (Epstein, 2011), and inferences are made from the premise as a chain of reasoning. Deductive reasoning supports inductive reasoning (Lee, Goodwin, & Johnson-Laird, 2008). Inductive reasoning, from a particular to the general, is supported by inferring general principles, that is, by deductive reasoning. Hence deduction supports the justification of a generalisation or conjecture through a logical argument. For example, a generalisation can be made in relation to the sum of two even numbers always being even, or that a zero is "added" (to the end of a number) when multiplying by ten, but justifying the generalisation, and making a claim about why it must be true (or not) for all numbers, requires a logical argument. Deduction goes beyond induction in not just recognising the pattern but in reasoning that, if it is true for one number, then it must be true for other numbers.

Mathematicians use both inductive and deductive reasoning and hence both are key to proof (Schoenfeld, 2009), but studies into mathematical reasoning often downplay the role of deductive reasoning. This downplay has two potential consequences. First, inductive reasoning is of no use in solving some problems; for example, a Sudoku puzzle (Lee et al., 2008). Second, if children are encouraged to rely on inductive reasoning, they may develop the belief that inductive arguments in mathematics are logically valid, and that a large number of numerical instances prove the generalization. This belief could interfere with older children and adults' ability to work with proof (Lee & Wheeler, 1987; Martin & Harel, 1989; Morris, 2007). Hence, it should be beneficial to develop ideas about mathematical justification compatible with the nature of mathematical proof from an early age, and that would include the development of deductive reasoning. Nevertheless, much research into young children's reasoning and justification has focused on the use of generalisations (e.g., Stylianou &

Blanton, 2011).

This paper focuses on a task intended to support deductive reasoning. As part of a teaching experiment to develop young children's learning in early arithmetic, a task was presented that required deductive reasoning. Devising such a task set challenges in ensuring that both the mathematics and the complexity of the logic were accessible to young children. In examining the way the children tackled the task the paper considers how the children were being inculcated into deductive reasoning and the relationship with their knowledge base in arithmetic.

Deductive reasoning and abstraction in mathematics

As indicated above, deductive reasoning refers to a chain of reasoning or, as McFeetors and Mason (2009) stated, the construction of "a sequence of moves, each building on the last leading to a desired endpoint" (p.286), where those moves are based on specific assertions. Reid (2002) further defined deductive reasoning as simple deductive reasoning and hypothetical deductive reasoning. Both seen as building blocks of proving these further definitions have clear distinctions. Simple deductive reasoning is enabled through problem-solving tasks as students assemble a chain of simple multi-steps from something that is known to be true. Reid gives an example of a problem posed from the children's story book *The Doorbell Rang* by Pat Hutchins and the way the teacher asked how the twelve cookies would be divided among four people. Reid described how one child predicted that each person would get three cookies. "Because three plus three would be, um, six, and another two threes would be six, and because three plus three is six, and another three plus three would be another six. So it's three." (Reid, 2002, p.235).

Reid's (2002) notion of hypothetical deductive reasoning relates to reasoning by establishing hypotheses. Reid gave an example of a child playing *Mastermind* and how the child reasoned three hypotheses related to the positioning of the blue, green and orange pegs in order to conclude that the position of the yellow peg must be correct. Reid proposed that, whilst simple deductive reasoning is a building block of proving, it is hypothetical deductive reasoning that is key to justifying the generalisations of inductive reasoning.

Whilst they both occur within practical activities, deductive and inductive reasoning require an abstraction or detachment from the practical activity. The abstraction of number or

reification of number, that is treating numbers as if they were physical world objects, is present in the simple multi-step deductive reasoning. In the example of the child's reasoning in *The Doorbell Rang* the child used the number of cookies as abstract cardinal values. In hypothetical deduction the reasoning is dynamic and transformational. In explaining a chain of reasoning there is a kind of abstract thought experiment. In creating hypotheses thinking is moving forwards and backwards (Simon, 1996). In the example given by Reid (2002), the child was looking back and reflecting on the positioning of the three pegs in order to look ahead and predict the position of the fourth peg.

In Reid's (2002) example of hypothetical deductive reasoning, the focus was on the position of the coloured pegs in *Mastermind* and was not connected with arithmetic. A question considered in this paper is how the dynamic reflective and predictive processes of hypothetical deductive reasoning might relate to reification of number. No longer are the cardinal values used in a simple chain of steps, but they are used in an abstract thought experiment in generating hypotheses.

One considered constraint of young children with a limited knowledge base in arithmetic is their reliance on counting strategies. A move to part-whole thinking requires abstract conceptual understanding and the ability to hold numbers as cardinal values as abstract reified numbers. The example given by Reid with the cookies demonstrated the child using part-whole thinking, where twelve was seen as being partitioned into three, four times (i.e., four groups of three). It raises a question about the possibility that tasks related to arithmetic which encourage hypothetical deductive reasoning have the potential to support concept reification.

Deductive reasoning in mathematics: a dialogic and linguistic perspective

From a Vygotskian social view, thinking and reasoning are influenced by cultural and historical factors (Luria, 1976); that is, thinking processes are "united within a cultural semiotic process of making sense of one's world, and of oneself" (Valsiner, 2007, p. 276). Within this sociocultural perspective, Dutilh Novaes (2013) views logic as "a culture-dependent phenomenon" and considered how deduction pertains to "cultural institutions and social phenomena in general" (p.465). Dutilh Novaes further proposed that, from a dialectic position, "a deduction is an *argument* (a piece of discourse), not an inner mental process"

(p.461). Hence deduction is viewed as a kind of language game that facilitates understanding of what is expected in reasoning analytically (Gee, 1996) and is informed by “socially accepted associations” (Forman, 1996, p. 131), where the quality of argument and logical coherence is culturally valued.

From this theoretical perspective the premise is that reasoning, as a mathematical process, is a cultural construct (Font, Godino, & Gallardo, 2013) and hence deduction, as a kind of dialogical and argumentative practice, arises from practice in the school classroom (Dutilh Novaes, 2013). For young children in their first few years of school, this socially accepted language game or discourse may be unfamiliar. Harris (2000) studied young children with limited schooling in pretence play situations, and found that children could reason from a set of make-believe premises with their play. However, young children may not have been enculturated into a habit of reasoning and systemised thinking in working with number problems. Furthermore, reasoning is a semantic process that depends on understanding the meaning of the premises, so consideration must be given to how the meaning of the premises will be understood.

Make-believe premises, in the sense of ‘*let’s pretend*,’ are established from beliefs or opinions, either the child’s or another person’s, in devising a play situation. Key to deductive reasoning is a willingness to dissociate from beliefs or opinions in arguing from a set of premises. A further key point is that a valid inference suggests there are no alternative conclusions consistent with the premises; there are only certain conclusions. Hence, the notion of necessity bridges reasoning from the premises to a conclusion. This is distinct from conclusions that may be based on beliefs or opinions. Furthermore, the notion of necessity is distinct from probabilistic reasoning where there are alternative possible conclusions. From these key points, three potential schema are proposed: belief/opinion schema of personal choice, probabilistic reasoning schema of alternative conclusions, and deductive reasoning schema with only certain conclusions. The third deductive reasoning schema then has two further distinctions as discussed above: simple deductive and hypothetical deductive reasoning.

As suggested earlier, deductive reasoning can be viewed as a language game. Hence an examination of the language used as the children engage in a task to support deduction may indicate transitions through these schema. Halliday and Matthiessen’s (2004) theory of

systemic functional linguistics (SFL), presents a pragmatic perspective where language has evolved because of its functions in making meaning out of a given environment. In this paper the children's discourse when working on a task is analysed according to their functional use of language, and in particular how the children use primary and modal tenses.

The primary tense, or actuality, expresses what is present at the time of speaking, for example '*it is*' or '*it isn't*.' Modality expresses the degree of likelihood or probability; for example, '*it might be*' and certainty; for example, '*it has to be*' or '*it cannot be*.' Modality is further divided into deontic and epistemic. Deontic modality indicates the necessity or possibility of acts, that is, socially regulated behaviour, and these are more commonly known to young children (for example, '*you have to sit still*' or '*you can't go out to play*'). Epistemic modality indicates the speaker's state of knowledge and degree of certainty (for example, '*it has to be*' or '*it can't be*'). The systemic use of modality by the speaker in deductive reasoning would indicate the notion of necessity, and the intention of this paper is that a study of the children's use of primary and modal tenses as functions of their language can be used to identify any changes from belief/opinion schema through to probabilistic and deductive schema.

Mathematics is typically seen as a subject based on facts and truths and so it is likely to have a high modality structure (Hodge & Kress, 1988). However, the necessity of mathematical statements is not always seen to be needed (Hodges, 2013). For example there is no need to state that two plus three must necessarily be five, the primary actuality tense is used instead as in two plus three is five.

In the examples given by Reid (2002), simple deductive reasoning from *The Doorbell Rang* problem, the child used the primary tense '*would be*,' and not modal terms such as '*could be*' or '*must be*.' In the conclusion, the primary tense was used again '*So it's three*.' However, in the example of hypothetical deductive reasoning in the game of *Mastermind*, the child used the modal term '*must be*' in determining the position of the fourth peg as the child expressed the necessity.

The study

This paper presents the analysis of a task to support deductive reasoning. The research method refers to the situated nature of the discourse (Radford, 2000) and analysis of dialogue

is related to micro-social contexts of the children's interactions with each other and with the task. The children's discourse is analysed in terms of the functions of their language, and in particular their use of primary tense and modality. McFeetors and Mason (2009) identified milestones in the early deductive reasoning of secondary students. An aim of the analysis is to identify any milestones with these young children and to consider how the use of language can help to define the milestones. Furthermore, in considering the children's learning in number, there is a question about how the dynamic looking back and reflecting in order to look ahead and predict might support the children's learning in number.

The three children engaged in the task presented in this paper, had been working with me as part of a larger study to develop young children's reasoning and collaboration in learning mathematics. I had been working with the class teacher in developing tasks for the children in her class over the school year. Prior to the task, the children had worked with me in finding the comparison relationship 'more or less than;' for example, 'two more than' or 'two less than.' Finding a number with, for example, two more or two less, relies on the comparison of two cardinal units and involves more than counting. The relational nature of numbers is abstract. Relationships between numbers do not refer directly to concrete objects; they can only be represented by concrete or symbolic objects (Steinbring, 2005). This abstract notion was seen as important in supporting the children's concept reification.

Structural dot patterns, including the patterns of a ten-frame (see Figure 1), were used to represent cardinal units for comparison, as a way to encourage part-whole thinking rather than counting. Studies on children's use of representations and structure have been shown to support part-whole thinking (Young-Loveridge, 2002). The use of the structured pattern of dots on the ten-frame was seen to enable the children to move from counting to a part-whole view of number based on subitising where they needed to.

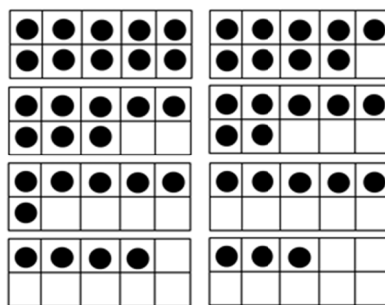


Figure 1. A set of ten frames representing 3 to 10

The task presented in this paper was developed as a puzzle with intrinsic logic (see Figure 2). In solving the puzzle, as in playing a game (van Oers, 2014), there are rules that restrict the position of the ten frames in completing the rectangle. The task was designed so that it did not relate to a system or pattern, but to deductive reasoning. To solve the puzzle, the children should complete a rectangle by placing ten-frame cards, which were more or less than the previous one, according to the condition recorded on the arrows around the rectangle (Figure 2). For example, the ten-frame following the initial 10 ten-frame could be either two more or two less than ten. Hence, the children had a choice in selecting either more or less, but only one of the choices in each case would give a correct solution and complete the puzzle (Figure 3). Eight ten-frame cards with values from three to ten were provided to complete the task. The rectangle was to be closed, meaning that the last ten-frame had to meet both the previous and the final conditions. As can be seen in Figure 3, the 9 ten-frame has been placed, so that it is three more than the 6 ten-frame and one less than the 10 ten-frame.

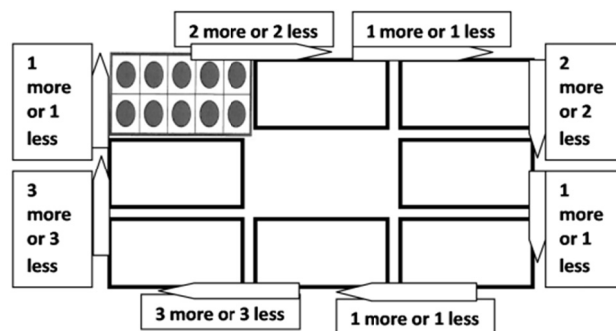


Figure 2. The More or Less task

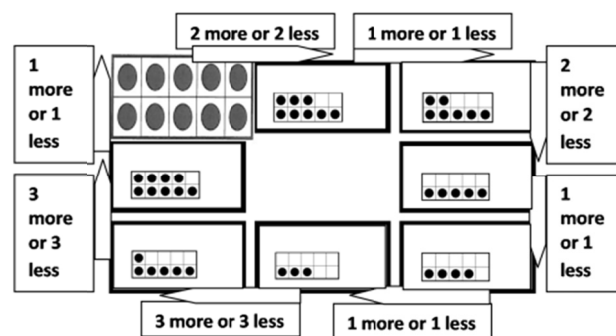


Figure 3. A completed version of the More or Less task

Framework for analysis

The children's interaction on the task was video recorded and transcribed. Analysis referred to both the transcriptions and the video recording for confirmation of the discourse in relation to the students' actions in completing the puzzle. The framework of de Freitas and Zolkower (2010) for the use of modality in classroom interaction was used to inform the discourse analysis.

- Actuality or primary tense ("this is what it is" or "this is what we are doing")
- Low level modularity questions ("what do you want?", "what shall we do?")
- Low level-deontic modularity ("we have to", "we need to")
- Medium modality ("I think...")
- High-level deontic modality ("we have to", "we need to")
- High-level epistemic modality ("it has to be", "it can't be")

These levels were related to the potential schema: belief/opinion, probabilistic reasoning, and deductive reasoning (simple and hypothetical) in the following way:

- Actuality or primary tense (solving comparison relationships): Simple deductive
- Low level modularity questions (a personal choice in what to do): Opinion/belief
- Low level-deontic modularity (proposal of what to do, not related to epistemic modality):
Opinion/belief
- Medium modality (hedging or conjectural speech act): probabilistic reasoning
- High-level deontic modality (necessity of action in relation to epistemic modality):
hypothetical deductive reasoning
- High-level epistemic modality (statement of necessity): hypothetical deductive reasoning

Results and analysis

The following section presents an extract from the video and analysis of what was happening during the extract.

Extract One: What shall we do?

I explained to the group of children that the rectangle had to be closed so the last ten-frame placed before the 10 ten-frame had to be one more or less than 10.

1. Helen: What shall we do, two more or two less? Two less, two less. *(Helen places the 8 ten-frame next to the 10 ten-frame.)*
2. Kim: One less. *(Kim reads from the next arrow.)*
3. Helen: It's eight. *(Helen counts the dots on the 8 ten-frame and Kim places the 7 ten-frame next.)*
4. Kim: Two more or two less? *(Kim reads from the next arrow.)*
5. Helen: Two more. *(Emma hands Helen the 9 ten-frame. Helen points to two dots on the 9 ten-frame.)* See there's two more. *(Helen places the 9 ten-frame.)*

Analysis of Extract One: What shall we do?

In transcript line 1 Helen asked "What shall we do?" This was an example of low level modularity question so reflected a belief/opinion scheme. In transcript line 4 Kim asked whether to use two more or two less, which could reflect deontic modality in relation to epistemic modality but Helen's response "See, there's two more" (line 5) related to actuality and simple deduction. Helen's statement "It's eight" (line 3) is further evidence of actuality and simple deduction.

When Helen selected the 8 ten-frame as two less than 10 (line 1), she identified the 8 ten-frame by subitising and identified the comparison relationship as a known fact. However, later (line 3) Helen counted all the dots on the 8 ten-frame. She then used this to determine seven as one less than eight, and indicating use of a counting strategy to identify the comparison relationship. Kim then identified and placed the ten frame without counting suggesting use of subitising. In transcript line 5, in identifying the ten frame with two more than seven, Helen points to the two dots on the nine card to show two more. It is possible that covering the two dots with her fingers helped her to see the seven left, or she may have examined the comparison relationship as part-whole. Emma then handed the 9 ten-frame without counting the dots, indicating that she identified the value through subitising.

Extract Two: We have to start again

6. Kim: We need to decide which one goes where. Do you want to do one more or one less?

7. Helen: One less, I mean one more

(The children look at the ten frames they had left (3, 4, 5, and 6) and Helen points to the 9 ten-frame and the next space on the rectangle.)

8. Helen: So that will be one less, it'll have to be one less.

9. Kim: It will have to be eight. No. We have to do one more, because we don't have....

(Kim looks at the 10 ten-frame) Oh no we have to start again.

(The children remove all the ten frames, apart from the 10 ten-frame, and start again.)

Analysis of Extract Two: We have to start again

Kim's statement in transcript line six "We need to decide" was related to the question "Do you want?" suggesting low level-deontic modularity and low level modality questions related to a belief or opinion. Helen's statement "That will be one less, it'll have to be one less" (line 8) suggested a transition from actuality primary tense to the realisation of necessity and epistemic modality. Kim continued to use the language of necessity in transcript line 9 and high-level deontic modality ("We have to do one more"). The children realised they had already used the 8 ten-frame and the 10 ten-frame so that neither of the options were open to them and that they will have to start again. The realisation that they had already used the two ten frames they needed meant they were reasoning dynamically. No longer were they carrying out a sequence of steps following more or less by choice, they were reflecting back on their previous choices.

Analysis of the use of ten frames suggests that no counting strategies were used. The children looked at the ten frames left and recognised that none of these was one more or one less than nine. They were aware of the comparison relationship one more or one less and they referred back to the 8 and 10 ten-frames already in position in the rectangle by subitising.

Extract Three: Ten is the biggest number

10. Helen: What about we do this – more, less, more, less, more, less *(Helen points to the*

spaces around the rectangle.)

11. Kim: But there's no bigger number, that's the biggest number. *(Kim pointed to the 10 ten-frame.)*
12. Kim: Why don't we do this one less, this one less, this one more, this one more, this one less, this one less? What do you want that to be? *(Kim points to the spaces around the rectangle and stops at the last closing space.)*
13. Emma: More.
14. Helen: No less, less.
15. Kim: That has to be nine. *(Kim continues to point to the 10 ten-frame.) (Helen places the 9 ten-frame in the space.)*

Analysis of Extract Three: Ten is the biggest number

In transcript lines 10 and 12 both Helen and Kim were asking for choices in how to complete the rectangle but there was a shift to medium modality, "What about we do?" and "Why don't we do?" suggesting an element of hedging or plausibility. This was interspersed with low-level modality questioning "What do you want that to be" (line 12). Kim also used actuality primary tense in declaring "That's the biggest number" (line 11) but this was accompanied by the phrase "But there's no bigger number" possibly indicating a move towards a conditional impact of the task. This was followed by Kim's high-level epistemic modality and expression of necessity "That has to be nine," hence a shift to hypothetical deduction.

The use of the ten frames in this extract is limited to the recognition of 10 as the biggest number and of nine as one less. Again there is no counting involved in this and Kim determines 9 as one less and Helen finds the 9 ten-frame without counting.

Extract Four: An incorrect solution

Helen had placed the 9 ten-frame in the space before the ten. The children then looked at the space after the 10 ten-frame and noted that they would need a ten frame two less than ten because ten is the biggest number. However, they then placed the 7 ten-frame rather than the

8 ten-frame. Following this, they placed the 8 ten-frame as one more than seven, and the 6 ten-frame as two less than eight. After that they completed placing the ten frames on the rectangle but they did not take into consideration the comparison relationships on the arrows and seem to place the ten frames randomly. Hence, they arrived at an incorrect solution.

Throughout this they do not count the ten frames. The 6 ten-frame is found in the comparison relationship two less than eight without counting. However they made an initial mistake in determining seven as two less than ten. It seemed they disregarded the conditions of the puzzle and returned to their own choices in which to put next as opinion/belief schema.

When they said they had finished, I asked why they had placed the 9 ten-frame before the 10 ten-frame.

16. Kim: Because ten's the biggest number we can't do one more, cos that would be 11 and we don't have 11.

17. Researcher: So why did you decide to put this one there? (*Researcher pointed to the 7 ten-frame next to the 10 ten-frame.*)

18. Kim: 'cause it's, what's this? Because it's...

19. Helen: No it's three less.

20. Kim: It's supposed to be two less. (*Kim swapped the 7 with the 8 ten-frame.*)

21. Researcher: (*Researcher pointed to the new position of the 7 ten-frame.*) Why wouldn't you put the 9 ten-frame there? You could have made that one more?

22. Kim: 'cause maybe you couldn't put anything here. (*Kim pointed to the 9 ten-frame before the 10 ten-frame.*)

23. Researcher: So where do you go after that? You need two more or two less. Can you use nine? (*Kim shook her head.*) So what you are going to have to use?

24. Kim: Five? (*Kim moved the five next to the 7 ten-frame but then moved it away and replaced it with the 4 ten-frame.*)

25. Helen: No you need the five, you need the five, you need the five there. (*Helen moves the 4 ten-frame away and replaces it with the 5 ten-frame.*)

Analysis of Extract Four: An incorrect solution

My questioning of the children had required them to demonstrate their reasoning. This further indicated Kim's use of high-level epistemic modality and hypothetical reasoning "We

can't do one more" (line 16). My questioning had also required them to reflect back on their decisions and realise their error in placing the 7 ten-frame instead of the 8 ten-frame, and they used actuality and present tense in confirming and correcting the error themselves. "What's this?" "It's three less" (lines 18 and 19). Kim's use of the word 'supposed,' as in "It's supposed to be" (line 20) suggests an obligation and hence modality in a deontic sense. However, the term suggested that the ten frame was required to be, or expected to be, which is not the same as "must be". In line 24 Kim's use of questioning and medium modality in suggesting five as a possibility, was then confirmed by Helen as high-level deontic modality relating to the necessity that the next ten-frame had to be five, there were no other options.

Extract Five: The correct solution

The children then worked independently but were unable to decide how to position the last three ten-frames. Emma had positioned the 3 ten-frame next to the 9 ten-frame. The comparison relationship between these two spaces was three more or three less, so it was possible that Emma had placed the three in relation to the value of three, and not the comparison relationship. Helen and Emma were getting restless and possibly frustrated, so I intervened further to point out the 3 ten-frame positioned next to the 9 ten-frame.

26. Researcher: Look at these two. This should be three more or three less than this one.

27. Kim: Those don't go together. *(Kim removes the 3 ten-frame)*

28. Researcher: Have another look

29. Kim: I think it's the six. *(Kim picks up the 6 ten-frame and counts the dots)*. Three less is six. *(Kim places the 6 ten-frame to cover the three dots on the 9 ten-frame)*. Three less is six. *(Emma places the 3 ten-frame instead of the 6 ten-frame)*.

30. Kim: I don't know if it is six.

As there was further restlessness from the children I intervened again.

31. Researcher: You think it's the six *(I place the 6 ten-frame next to the 9 ten-frame)*. So see if that works. So you've got the last two to choose which way they go around.

32. Kim *(points to the 5 ten-frame)* One less is four *(and places the 4 ten-frame)*. That's one more or one less *(Emma points to the last place and Kim places the 3 ten-frame)*.
Done

33. Emma: We're all done.

Analysis of Extract Five: The correct solution

The statements used suggested actuality and primary tense as the children checked the comparison relationships “Those don’t go together” (line 27) and medium modality in the use of hedging terms “I think it’s the six” (line 29) “I don’t know if it is the six” (line 30). Actuality or primary tense was then used by Kim in stating “One less is four” and “That’s one more” (line 32).

Kim used counting to determine the number of dots on the 6 ten-frame but her placement of the 6 ten-frame to cover three dots on the 9 ten-frame suggested she may have been using a visual comparison of the dots and subitising. The identification of ten frames for the comparison relationships one more or less was carried out without counting. Although completed correctly the children did not check the comparison relationship for three more or less between the 3 ten-frame and the next 6 ten-frame, so they did not fully appreciate if the rectangle was closed at this point.

Discussion

Whilst the children were able to find a solution, they were not able to do this independently. It was not clear how much this was due to a limited knowledge base and how much might have been due to the complexity of the logic required for the task and their early use of deductive reasoning.

Their use of the ten frames in finding the comparison relationships suggested some reliance on counting. The comparison relationship one more or one less was mostly known to the children, but even then, on one occasion, Helen counted the 8 ten-frame (line 3) to determine how many dots there were in order to find seven as one less. The ten frames were used to visually determine some other comparison relationships. For example, Helen pointed to the two dots to show nine as two more than seven and Kim found five as two less than seven without counting, but even so she appeared to be hedging this decision in questioning the five (line 25). Later Kim determined six as three less than nine by covering three of the dots on the 9 ten-frame with the 6 ten-frame as a way of seeing the difference of three (line 29). Even so

she had counted the six dots on the 6 ten-frame first and was still hesitant that was the solution.

The children made an error in placing the 7 ten-frame as two less than ten (Extract Four). Emma had positioned the 3 ten-frame as being three less than nine, perhaps focusing on the cardinal value of three rather than on the three as the difference. Nevertheless when asked to re-focus on these errors they were rectified by the children. Kim's hesitation in placing the 5 ten-frame was confirmed by Helen. Kim's hesitation with placing the 6 ten-frame as three less than nine was not supported by the children. The children also did not regard the other three more/less relationship in placing their final ten-frame (lines 32 and 33) so it is not clear if they were able to determine a difference of three.

A lack of fluency in using the comparison relationships may have hindered the children's opportunity in correctly completing the rectangle independently. However, the ten frames did provide an opportunity for the children to count or cover dots and this possibly supported them more than if the cards had been numerals. The covering of the dots also suggested some shifts to part-whole thinking as the children saw nine as six and three, as well as seeing seven as two and five, and nine as two and seven.

Analysis of the function of the language indicated that the children were able to use modal terms to suggest necessity at times. The children started with low-level modality questioning and low-level deontic modality in placing the ten frames. Actuality and the primary tense was used to support the positioning of ten frames. It was not until the children reached the dilemma in realising they had used both the 8 and 10 ten-frames (Extract Two) that there was an initial move to high-level epistemic and deontic modality. This may have indicated a first milestone in early hypothetical deductive reasoning but they were unable to move backwards and transform the previous steps, so their solution was to remove all cards and to start again.

The process carried out by both Kim and Helen in pointing around the rectangle may have instigated dynamic and transformational thinking, as Kim looked to solve the puzzle by starting at the position before the 10 ten-frame (Extract Three). At this point it seemed Kim was able to develop a chain of reasoning that required looking back and reflecting in order to look ahead and predict the only possible next ten-frame. Kim was working with imagined actions on objects, and treating the comparison relationships as objects to be reasoned with in relationship to the premises. She had established a set of hypotheses which required holding a chain of comparison relationships as dynamic images.

Such dynamic imaging seemed related to the thinking of the child completing the *Mastermind* game, rather than to the child using simple multi-step deduction to solve the problem of repeated addition with the cookies (Reid, 2002). This deductive thinking was reflected in the language as Kim used terms such as “That has to be one less” and “It can’t be one more.” Unlike the simple multi-step deduction of a calculation where the necessity does not need to be stated, Kim was stressing the certainty and necessity that had been deduced from the premises. Even though Kim appeared to move to high-level modality, the next extracts suggest evidence of plausibility and hedging until the children reverted to choosing ten frames to place and no longer looked at the conditions.

It is not clear whether the lack of fluency of the children in their knowledge base or the complexity of the puzzle hindered the children in completing the task independently. As considered earlier from a dialogic perspective, deduction as a semantic process depends on understanding the meaning of the premises. Whilst it seemed the children realised the rectangle had to be closed at the point before the 10 ten-frame, and so realised the meaning that the last ten-frame had to meet both the previous and the final conditions, they may not have been able to hold this meaning continuously around the rest of the rectangle. It is possible that the children were not used to such conditions being stipulated and were more used to finding systems and patterns to follow. The nature of such conditions as arbitrary premises could seem regimented to children and hence not follow the normal repertoire of dialogical interactions (Dutilh Novaes, 2013). In cooperating together in testing a set of premises and hence the validity of deduction, such experiences could help children reconceptualise deductive reasoning.

Concluding remarks

The extracts analysed here were from the children’s first attempt with the puzzle. Nevertheless the evidence suggests some potential in using such tasks to encourage deductive reasoning with young children. The children, at least Kim in particular, used both high-level deontic and epistemic modality, and this is consistent with other findings (Byrnes & Duff, 1989; Papafragou & Ozturk, 2007). The children were using modality in relation to a chain of

reasoning and hence deduction. Further research is needed to determine how longer term use of such puzzles might impact on young children's learning in early arithmetic, particularly in supporting concept reification for cardinality and hence part-whole thinking. Furthermore, research is needed to determine how such tasks could impact on young children's algebraic thinking in supporting a level of deductive reasoning required to support the justification of generalisations.

The analysis of the language indicated a level of authority by the children in validating their thinking (Hodge & Kress, 1993). The children had been able to go beyond opinion and the use of actuality and primary tense to the modal level of reasoning necessity. Not only could such tasks help children learn to reason deductively as a prerequisite for formal studies of proof, the use of such tasks could help young children perceive possibilities and certainties and become more aware of their own mental state when engaging in mathematics as a subject with a high modality structure.

References

- Battista, M. T., & Clements, D. H. (1995). Geometry and proof. *The Mathematics Teacher*, 88(1), 48-54.
- Byrnes, J., & Duff, M. (1989). Young children's comprehension of modal expressions. *Cognitive Development*, 4, 369-387.
- de Freitas, E., & Zolkower, B. (2010). *Discursive authority in the mathematics classroom: Developing teacher capacity to analyze interactions in terms of modality and modulation*. Paper presented at the Mathematics Education and Society 6 Proceedings, Berlin, Germany.
- Diezmann, C. M., Watters, J. J., & English, L. D. (2002). Teacher behaviours that influence young children's reasoning. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings 27th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 289-296), Norwich, UK.
- Dutilh Novaes, C. (2013). A dialogical account of deductive reasoning as a case study for how culture shapes cognition. *Journal of Cognition and Culture*, 13, 459-482.

- Epstein, R. L. (2011). *Reasoning in science and mathematics: Essays on logic as the art of reasoning well*. Socorro, NM: Advanced Reasoning Forum.
- Font, V., Godino, J., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics*, 82, 97-124.
- Forman, E. (1996). Learning mathematics as participation in classroom practice: Implications of sociocultural theory for educational reform. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 115-130). Mahwah, NJ: Lawrence Erlbaum Associates.
- Gee, J. P. (1996). *Social linguistics and literacies: Ideology in Discourses* (2nd ed.). London: Taylor & Francis Ltd.
- Halliday, M. A. K., & Matthiessen, C. (2004). *An introduction to functional grammar* (3rd ed.). London: Arnold.
- Harris, P. I. (2000). *The work of the imagination*. New York: Wiley.
- Hodge, R., & Kress, G. (1988). *Social semiotics*. Cambridge, UK: Polity Press.
- Hodge, R., & Kress, G. (1993). *Language as ideology* (2nd ed.). London, UK: Routledge & Kegan Paul.
- Hodges, W. (2013). Modality in mathematics. *Logique et Analyse*, 56(221), 5-23.
- Johnson-Laird, P. (2008). Mental models and deductive reasoning. In J. Adler & I. Rips (Eds.), *Reasoning: Studies of human inference and its foundations* (pp. 206-222). Cambridge, UK: Cambridge University Press.
- Lee, L., & Wheeler, D. (1987). Algebraic thinking in high school students: their conceptions of generalisation and justification: Research report. Montreal, Concordia University.
- Lee, L. N. Y., Goodwin, G. P., & Johnson-Laird, P. N. (2008). The psychological puzzle of Sudoku. *Thinking & Reasoning*, 14, 342-364.
- Luria, A. R. (1976). *Cognitive development: Its social and cultural foundations*. Cambridge, MA: Harvard University Press.
- Markovits, H. (2013). The development of abstract conditional reasoning. In P. Barrouillet & C. Gauffroy (Eds.), *The development of thinking and reasoning* (pp. 71-91) New York, NY: Psychology Press.
- Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, 20, 41-51.

- McFeetors, P. J., & Mason, R. T. (2009). Learning deductive reasoning through games of logic. *The Mathematics Teacher*, 103, 284-290.
- Morris, A. K. (2007). Factors affecting preservice teachers' evaluation of the validity of students' mathematical arguments in classroom contexts. *Cognition & Instruction*, 25, 479-522.
- Papafragou, A., & Ozturk, O. (2007). *Children's acquisition of epistemic modality*. Paper presented at the Proceedings of the 2nd Conference on Generative Approaches to Language Acquisition North America (GALANA), Somerville, MA.
- Piaget, J., & Inhelder, B. (1969). *The psychology of the child*. New York, NY: Basic Books.
- Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics*, 42, 237-268.
- Reid, D. A. (2002). Describing reasoning in early elementary school mathematics. *Teaching Children Mathematics*, 9, 234-237.
- Schoenfeld, A. H. (2009). Series editor's foreword: The soul of mathematics. In D. A. Stylianou, M. L. Blanton, & E. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. xii-xvi). New York: Routledge.
- Simon, M. A. (1996). Beyond inductive and deductive reasoning: The search for sense of self. *Educational Studies in Mathematics*, 30, 197-210.
- Steinbring, H. (2005). *The construction of new mathematical knowledge in classroom interaction: an epistemological perspective*. New York: Springer.
- Stylianidis, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, 38, 289-321.
- Stylianou, D. A., & Blanton, M. L. (2011). Developing students' capacity for constructing proofs through discourse. *Mathematics Teacher*, 105, 140-145.
- Valsiner, J. (2007). *Culture in minds and societies*. New Delhi, India: Sage.
- van Oers, B. (2014). *The roots of mathematising in young children's play*. New York: Springer.
- Young-Loveridge, J. (2002). Early childhood and numeracy: Building an understanding of part-whole relationships. *Australian Journal of Early Childhood*, 27(4), 36-42.