

# **The Role of Units in the Concept of Length for Four- to Six-year-old Children**

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## **Abstract**

The understanding of the unit is a key point in the development of the concept of length. The understanding of unit is necessary for measuring as well as for comparing indirectly by using unit iteration. This article describes a study that investigated the use of units by children when they were comparing lines indirectly. The relationships between different aspects of the use of units are examined, including the understanding of units, and the use of units for decomposing and for counting.

**Keywords :** Kindergarten, preschool, measurement, length, linear comparison, units, mathematics

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## Introduction

Castagnetti and Vecchi (2002, p. 14) claim that, “dealing with measures is the most suitable way to introduce five-year-old children to the world of figures and mathematical terms”. Of all the measurement attributes, length is the most important because it is the most primary attribute (Buys & Moor, 2008, p. 18). This claim is based on the fact that the concept of length is fundamental to the development of other magnitudes such as area and volume. Also many measurement tools use length for their scale; for example, kitchen scales use linear scale to indicate the total mass.

Lorenz (2004) argued that length plays a fundamental role not only for different dimensions of measurement but also for arithmetic. In addition, Lorenz cited studies by Carter (1983) which revealed that many people represent numbers mentally in a linear way like a “mental number line”. The unitary nature of measurement is also seen in counting where each object is seen as a countable unit. In working with fractions, the understanding of unit and partitioning of the unit is fundamental. For understanding place value, units are a central idea where single units are combined to build composite units such as 10s, 100s, etc.

The importance of knowledge about length can also be seen in many curricula. For example, in the German and Australian curriculum documents, length is the dimension of measurement that is mentioned most frequently and specified most precisely. However, there are still relatively few empirical studies investigating in detail the development of younger children’s ideas about length (Gasteiger, 2010).

For many children, the earliest length experiences involve the direct comparison of lengths. When comparison needs to be made indirectly, the use of tools becomes necessary and the importance of units come to the fore (Zöllner & Benz, 2013). The understanding of unit and unit iteration is necessary for both measuring and comparing, which will be explained further in the theoretical background. Bragg and Outhred (2001, p. 215) assert that, in order to help children to build a broad understanding of units, “researchers and teachers need to work together to investigate how to teach young students to link experiences with informal units with formal measurement.” This paper contributes to this agenda by reporting on which competencies concerning units 4- to 6-year-old children show, so that teachers can connect to these.

### ***Theoretical Background***

On the first view comparing and measuring seem to be very different processes. When two lengths are compared the result is often expressed in a qualitative way; for example, “this line is longer than the other line”. When measuring normally, the result is a quantitative result; such as, “this line is as long as 15 sticks” or “that town is 2 km away”.

If we look at different possibilities of comparing lengths, we can find many similarities between comparing and measuring. If lines cannot be compared directly or if there is no tool available that is large enough for the indirect comparison, units and also unit iteration are needed. When unit iteration is used for the indirect comparison, first the lines must be decomposed into units and then the quantity of units can be compared. The quantity of units can be compared through one-to-one correspondence or through counting.

In order to compare quantities, first a measuring process with (standardised) units for each line has to be conducted. We can see that both in the process of comparing indirectly with shorter media and in process of measuring three main aspects can be found: (1) choosing a unit; (2) decomposing the length into equal parts (procedural activity: unit iteration); and (3) counting the units.

These aspects are used with slightly different terms when different authors describe the measuring concept (eg. Benz, Peter-Koop & Grüßing, 2014; Nunes, Light, & Mason, 1993; Nührenbörger, 2002; Stephan & Clements, 2003). For all three aspects, the unit plays an important role. Many years ago, Piaget and Inhelder (1974) pointed out the importance of the unit. They described how through using unit and unit iteration, one can create any length one wants so every magnitude can be built. An understanding of the unit is, therefore, a central aspect of competence in the development of the concept of length. Many other authors point out how an understanding of the unit is essential for the development of a measuring concept (e.g., Boulton-Lewis, Wilss, & Mutch, 1996; Bragg & Outhred, 2001; Hiebert, 1981; Lehrer, 2003; Peter-Koop & Nührenbörger, 2011; Schmidt & Weiser, 1986)

Piaget and Inhelder (1974) suggested that although children can conduct some kind of unit iteration, often they have not developed an understanding of the unit. Piaget observed that children could use steps for measuring a length when they were asked to do so, but they did not pay attention to whether their steps were same length. Piaget and Inhelder concluded that

the use of units to decompose a length requires an understanding of transitivity<sup>1</sup> (see also, Steffe & Hierstein, 1976). In Piaget's view, children cannot develop an understanding of a metric unit until the age of 8 years (Piaget & Inhelder, 1974). Bragg & Outhred (2001) have also pointed out that children in primary school do not pay attention to units.

Freudenthal (1983) raised concerns about the idea that for many measuring processes it is not sufficient to choose only *one* unit. Often one unit will not be sufficient to measure a length in equal parts exactly as there can still be a part of the object to be measured that remains, and this remaining length will be shorter than the chosen unit. For this remaining length, a further measuring process is needed and a new unit has to be chosen. Then the entire object to be measured can be matched to the original units together with smaller units. Then all the units are counted to determine the result; for example, "the table is 15 sticks and three matches long".

Therefore, the understanding of unit consists also of the awareness that for different lengths, different units can be chosen. If different units are used, the parts of the lengths that are separated with different units must be considered separately.

The importance of developing students' awareness of mathematical pattern and structure (AMPS) has been emphasized in the work of Mulligan (2011), who has shown how measurement involves iterating equal-sized units in space. Mulligan's research has shown that an intervention program can help to draw students' attention to shape, size, spacing, and unit of repeat – critical attributes for measurement.

As we now can see the understanding of the unit is important. One demand for teachers is to support children in the development of this understanding. Teachers need therefore to know what competencies concerning units, children at the age of 4-6 years show, so that they can capitalise on this understanding. This study was guided by two research questions; namely: *What understanding and use of length units do 4- to 6-year-old children demonstrate?* and, *To what extent is children's expertise with counting of units and unit iteration interdependent?*

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<sup>1</sup> Piaget referred in this context to the equivalence aspect of transitivity. If  $a$ ,  $b$  und  $c$  are lengths, then if  $a=b$  and  $b=c$ , it can be concluded that  $a=c$ .

### ***Study Design***

In order to investigate children's concept of measuring, 40 children aged 4 to 6 years were interviewed individually with tasks designed to focus on the three aspects of length measurement described earlier; namely, choosing a unit, decomposing a length into equal parts (procedural activity: unit iteration), and counting the units.

The individual interview was constructed as a task-based clinical interview where children could also use materials to solve problems. Each child was presented with the same tasks in the same order. This standardisation allowed quantitative analysis of the children's responses. However, although the tasks were introduced to the child in the same way, the child could respond individually to each task. There were no criteria that the interview had to be finished if a task could not be solved. Each interview was video recorded so that the responses of the children could be analysed in detail at a later time.

Due to space constraints, the results and discussion presented here focus on four tasks from the larger interview, where children were asked to compare two lengths in four different ways. The lines were always arranged as shown in Figure 1, so it was not possible for the child to conduct a direct comparison. The four tasks were not presented one after the other, there were other tasks between them.

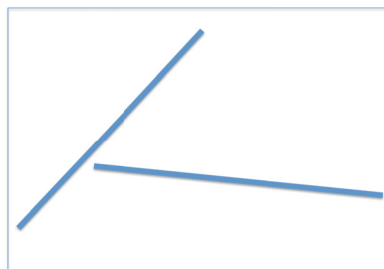


*Figure 1.* The arrangement of the lines

In the first task, children could use many tools of equal length (matches) to compare the two lines indirectly. One line was seven matches (31.5cm) in length and the other line was eight matches (36 cm) long. The children were offered more than 20 matches, so that they had enough matches to lay the matches on both lines.

In the second task, they were offered only one tool (one match). One line was five matches long (22.5cm) and the other line was six matches long (27cm). The shorter tool was offered so that it could be used multiple times for decomposing the length and unit iteration could be applied.

In the third task, the children were not given any tool for the comparison.

In the fourth task, a range of different tools was offered. The children could choose from standardized and non-standardized tools; for example, wooden sticks, different cords, different rulers, and a measuring tape. Some tools were shorter than the lines and some tools were longer than the lines. In the third and the fourth tasks, the lines were 1.20m and 1.30m in length.

## Results

Analysis of the videotaped interviews using qualitative content analysis (Mayring, 2001) resulted in the generation of different categories to code the children's processes (Kluge, 2000) and conclusions drawn about the children's conceptual understanding of units. First each task was analysed separately.

### ***Task one: Comparing two lines with many tools of equal length (matches)***

Figure 2 shows diagrammatically two categories of response to the task and sub-categories of each. Children either repeated the unit by placing matches on the line (shown in solid bubbles) or they did not repeat the unit (shown in dotted bubbles). The children who did not repeat the unit will not be discussed further here. Of the 40 children interviewed, 24 children placed matches on one line. Nineteen of these children also placed matches on the second line. By analysing the solution paths of the 19 children, it was evident that 12 children placed the matches without gaps and overlaps, and seven children did not pay attention to gaps or overlaps. Although 19 children placed matches on both lines, only 16 of them compared the quantities of matches and came to a result.

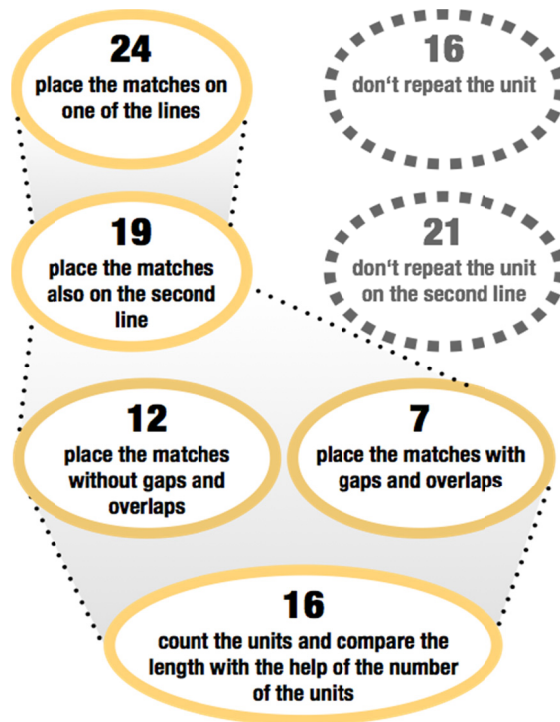


Figure 2. Results of the indirect comparison with many tools of equal length

### *Elsa*

(aged 6 years three months) was the only child who used unit iteration successfully, but she did not use counting for comparing. She first placed matches on the shorter line and made sure that there were no gaps and overlaps. Without counting, she took all matches that were lying on the first line and placed them on the second line, then said:

Elsa: This one is longer (pointing to the longer line).

Interviewer: Why is that longer? How do you know that?

Elsa: Because I have taken there (pointing to the longer line) as many matches as there (pointing to the shorter line).

(Without saying anything else, she pointed to the [remaining] part of the longer line without matches).

Elsa used the same number of matches for comparing both lines and remarked, because she could see on the second line that there was still some part of the line which was not covered with matches, she concluded that the second line had to be longer. It is important to note that

we cannot conclude from this action that Elsa had an understanding of unit. She knew that the sum of all partial lengths must be the same if the lines are equal. But we do not know if she was aware that the partial lengths must have all been of equal length.

We can conclude from the analysis of the first task: Many children are familiar with the procedure, they use units in order to divide the objects into smaller parts of the same length and then count the units and compare the number of the units. However, it cannot be said that all of these children have an understanding of units.

***Task two: Comparing two lines with a non-standardized tool, shorter than both lines***

When the children were offered only one tool, fewer children showed a response ( $n=12$ ) that could be linked to units, as shown in Figure 3. Only five children tried with the help of one match to decompose both lines in units. But no child did this in an accurate way. For every child, the actions consisted gaps and overlaps. The other seven children who did not use the match, counted “imaginary units”. They counted along both lines and compared the result of their counting. However, they did not pay attention to the need to have units of equal length. All 12 children who showed in this task an activity that could be linked to units, placed matches on both lines.

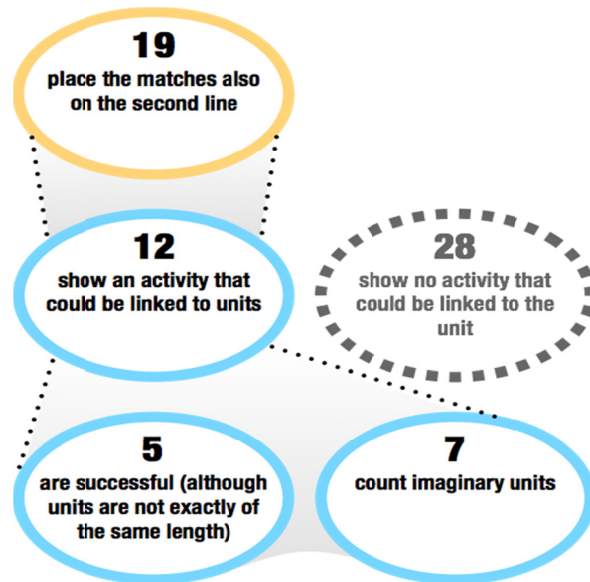


Figure 3. Results of the indirect comparison with one tool

The results can be interpreted as showing that only children who were able to conduct the unit iteration with many tools were also able to do this with one tool. Nevertheless, many of the children who were able to conduct the unit iteration with many tools were not able to transfer the idea of the units and apply this idea using only one tool. However, they were able to conduct the idea of decomposing and counting.

***Task three: Comparing two lines without any tools***

In Task 3, the results of which are shown in Figure 4, only four children responded in a way that could be linked to the unit. These four children applied units by counting imagined units. They counted both lines but they did not pay attention to parts of equal length. The pictures in Figure 5 show how Samuel counted imagined units. He used his feet to create “steps” along each line.

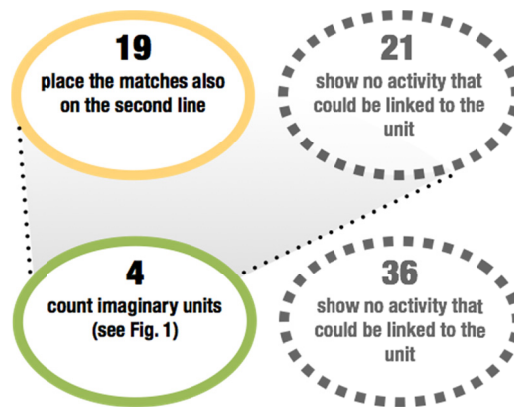


Figure 4. Results of the indirect comparison without any tool

***Samuel***

(aged 5 years, 11 months) solved the task in this way:

Interviewer: What do you think, which line is longer?

Samuel: (made something like steps on the lines) This one (1.30m) is a little bit longer.

Interviewer: How did you find that out?

Samuel: I measured it with steps.

Interviewer: Can you show me again?

Samuel: (made steps on one line and counted while doing it) 1, 2, 3, 4 (on the other line he also counted) 1, 2, 3, 4. Äh, equal length.

Interviewer: But earlier, you said that one is longer.

Samuel: Yes, because I did not start at the beginning.



Figure 5. Samuel (aged 5 years, 11 months) counting imaginary units

Samuel said, that he “measured with steps”. He came to the result by decomposing both lines with the help of steps. When he was asked to explain his solution path, contrary to his first attempt, he got the result that both lines were the same length. Therefore, he thought about his first try and came to the conclusion that on one line he did not start exactly on the end of the line. This showed his awareness of the need to measure exactly from the origin to the end of an object. While Samuel had correctly used matches iteratively in the Task one without any gaps or overlaps when he was given a common unit, he had a very casual attitude to the consistency of the size of his step units. It seems that he did not pay attention in this case to the need for equal units or he was unable to create equal units to use.

Samuel seems to be aware of the following facts:

- It is necessary to decompose both lengths;
- It is reasonable to count “units” and to compare the quantity of the units;
- It is important to start exactly on the end with counting and to finish counting exactly on the other end.

However, Samuel did not seem to be aware that “units” must always have the same length. Therefore it was concluded that even the children who were able to conduct the unit iteration with many given tools seemed not to use their idea about units in every situation.

**Task 4: Comparing two lines with one of a range of different tools**

When children were offered many different tools, 35 children choose a standardized measuring tool (see Figure 6). This similar to the findings of other researchers showing that when children are asked to do an indirect comparison, most children choose standardised tools, if they were available (e.g., Nunes et al., 1993).

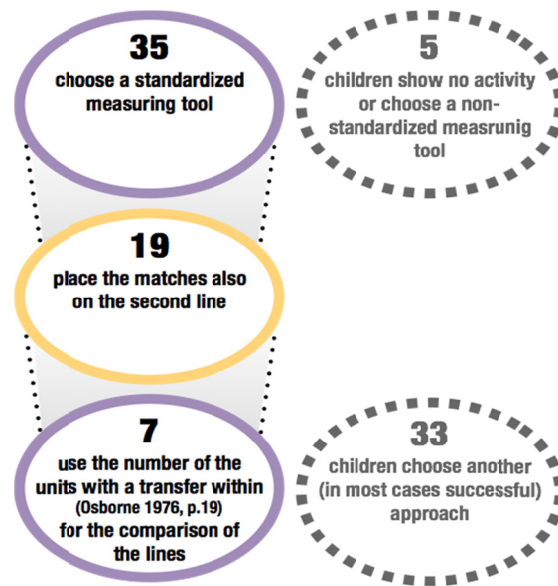


Figure 6. Results of the indirect comparison without any tools.

Probably the children imitated what they had seen when adults wanted to know how long a line is and so they used a measuring process with standardised tools. Standardised tools could attract children because they are not often easy available like measuring sticks. Another reason for the preference of the standardised tool could be that it seems easier to handle it. But if they use the standardised measuring tool it is not connected with the application of units.

**Alfonso**

(aged 5 years eight months) was one of the children who used a standardised measuring tool, but did not show a response that showed an understanding of units. He chose the folding builders' ruler. He opened it so that it had the length of 120cm and looked to see if the opened

ruler and the line were the same length. Then he paid attention to the second line. He laid down the builders' ruler on the second line and compared the two lengths. Alfonso did not show any understanding of units in his response as he effectively used the ruler like a stick: he directly compared the ruler to the lines, he matched the length of the first line to the ruler and then compared the second line to the ruler.

Only seven of the 35 children who chose a standardised tool referred to units. The majority of them judged line length on the basis of the comparison of numbers. Hence, they used a standardised tool and noticed the numbers on the scale. They either read the number or asked that the numbers were read to them. They then used the order of the numbers to compare the lengths of the lines.

### ***Luka***

(aged 5 years four months) belonged to this group of children who compared line lengths by ordering the numbers. She put a measuring tape on the first line and asked the interviewer to read the number that corresponded to the endpoint of the line. She was told that it was 120. On the second line, she also used the measuring tape and asked for the number to be read to her. She was told that it was 130. Then she asked which of these numbers was larger because she knew that 120 and 130 were beyond her number-reading skills. However, it became clear that the numbers were not beyond her active understanding of numbers. When she was told that 130 was the larger number, it was obvious to her that the line that was named 130 was therefore longer. Luka seemed to transfer the idea of the relative magnitude of the two numerical quantities to the relative magnitude of the lengths. This has been referred to in the literature as "transfer within" - Luka transferred the order relations from number to length (Schmidt and Weiser, 1986, p. 125).

This led to the following hypothesis: Because all seven children who chose a standardised tool and used the number of the units for a transfer within for the comparison of the lines it could be assumed that there is a connection between these two processes. Nevertheless, it must be remembered that 12 children used unit iteration but did not use the standardised tool for a measuring activity. Therefore, it can be concluded that an idea about unit iteration, or respectively an idea about the units does not compulsorily go with the usage of a measuring tool.

### Connections between the tasks

In this section of the paper, the categorised responses of the children across three of the four tasks are described (see Figure 7). The reason for doing this is to build more detailed pictures of the children's knowledge. For example, Samuel's responses to Task one and Task three taken together show the complexity of his developing awareness of units of length. Task four was not included in this figure as it was analysed in an earlier section of the paper.

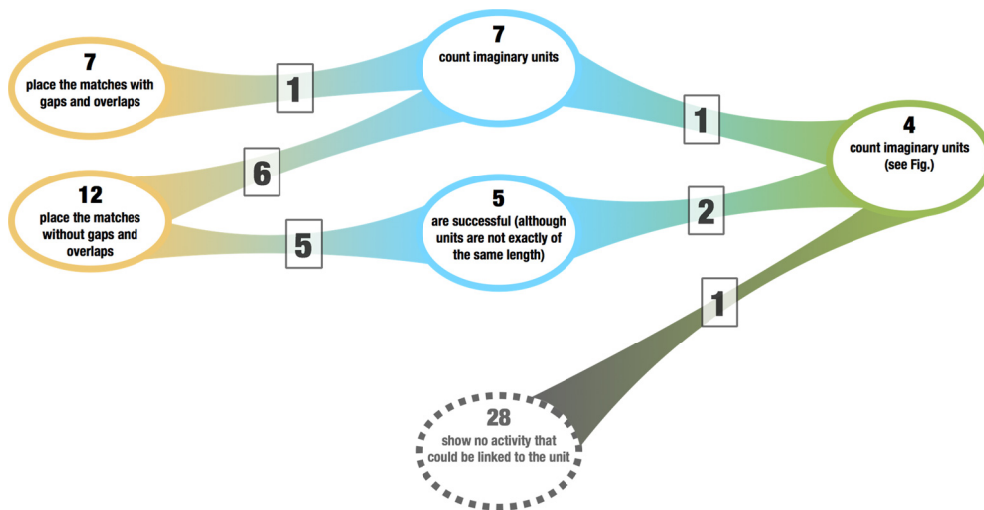


Figure 7. Relationships among different tasks (Left: Task one; Centre: Task two; Right: Task three).

In Figure 7, the bubbles on the left show categorised responses to Task one, those in the centre are categorised responses to Task two, and those on the right refer to Task three. The numbers on the lines between two bubbles show how many children belonged to both groups.

In Figure 7, first the dependency between the unit iteration with many tools and with one tool is presented. Only children who could conduct the unit iteration with many tools without gaps and overlaps (Task one – left hand bubble) also used one tool as a unit for unit iteration and came to a result (seven children in the centre bubble). This could be lead to the conclusion that already the idea of a unit had been noticed. But these children also did not pay attention to conducting the unit iteration without gaps and overlaps when offered only one tool. Also only children who could use unit iteration with many tools counted imaginary units

(five children in the centre bubble). They seemed familiar with the idea of decomposing the line and counting the decomposed parts, but did not pay attention to the length of the decomposed parts.

Looking at the dependency between the situation when only one or no tool is offered, first it has to be noticed that only four children showed an activity, which can be connected with units if no tool was available (right hand bubble). They counted imaginary units. Interestingly, for these four children there cannot be concluded a dependency to the solution paths when offered only one tool for comparing indirectly. One child counted imaginary units, two children used unit iteration with gaps and overlaps and one child did not show an activity, which can be connected with an activity including units. This one child also showed no activity, which could be connected to the idea of units. This observation confirms that counting imaginary units without any tool is not an indicator that units will be used for the solution of tasks with other conditions. Therefore it can be concluded that decomposing and counting are familiar ideas but not the idea of decomposition in equal parts resp. in units.

## Discussion and Conclusion

The analysis of the interview data revealed that many children are able to use unit iteration in a correct way without any gaps or overlaps when they are offered many real objects of the same kind as —unit and also counted their —units. When these children are offered only one object of each standardised or non-standardised shorter tool, however, they either count with some kind of imaginary units while holding the tool in their hands, or they decompose the length into different parts, without producing equal partial lengths.

The results can be interpreted in various ways. On the one hand, it could be inferred that children at this age have no technical skills to find a solution when there are not many —units available. On the other hand this could be seen as an indicator that although children are capable of conducting unit iteration with many tools correctly on a procedural level, having an idea about units seems to be another part of a conceptual knowledge.

The idea of decomposing and counting the decomposed parts can be observed more often than the idea of the need of using units of the same length. This may lead to the conclusion

that a conceptual knowledge of units is quite demanding which could be explained through further empirical and theoretical analysis.

The implication of these results for teaching is that focusing only on unit iteration is not enough to develop a comprehensive understanding of unit. If teachers only give children experiences of iterating with common informal units it does not necessarily lead to more than procedural unit iteration. One possibility to support an understanding of the unit could be, for example, to discuss “unit” iteration with unequal units, so that children are encouraged to think about the necessity of decomposing in equal partial lengths.

A discussion about these kinds of problems can lead to a deeper, connected, meaningful understanding of length and the need for standardised units to measure accurately.

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